

Kennesaw State University DigitalCommons@Kennesaw State University

Faculty Publications

9-1-2014

Fibonacci Number of the Tadpole Graph

Joe DeMaio

Kennesaw State University, jdemoio@kennesaw.edu

John Jacobson

Moxie, johnjacobson8128@gmail.com

Follow this and additional works at: <http://digitalcommons.kennesaw.edu/facpubs>



Part of the [Discrete Mathematics and Combinatorics Commons](#)

Recommended Citation

DeMaio, Joe and Jacobson, John, "Fibonacci Number of the Tadpole Graph" (2014). *Faculty Publications*. 3654.
<http://digitalcommons.kennesaw.edu/facpubs/3654>

This Article is brought to you for free and open access by DigitalCommons@Kennesaw State University. It has been accepted for inclusion in Faculty Publications by an authorized administrator of DigitalCommons@Kennesaw State University. For more information, please contact digitalcommons@kennesaw.edu.



Fibonacci number of the tadpole graph

Joe DeMaio^a, John Jacobson^b

^a*Department of Mathematics,
Kennesaw State University, USA*

^b*Moxie, Atlanta, Georgia, USA*

jdemaio@kennesaw.edu, johnjacobson8128@gmail.com

Abstract

In 1982, Prodinger and Tichy defined the Fibonacci number of a graph G to be the number of independent sets of the graph G . They did so since the Fibonacci number of the path graph P_n is the Fibonacci number F_{n+2} and the Fibonacci number of the cycle graph C_n is the Lucas number L_n . The tadpole graph $T_{n,k}$ is the graph created by concatenating C_n and P_k with an edge from any vertex of C_n to a pendant of P_k for integers $n = 3$ and $k = 0$. This paper establishes formulae and identities for the Fibonacci number of the tadpole graph via algebraic and combinatorial methods.

Keywords: independent sets; Fibonacci sequence; cycles; paths

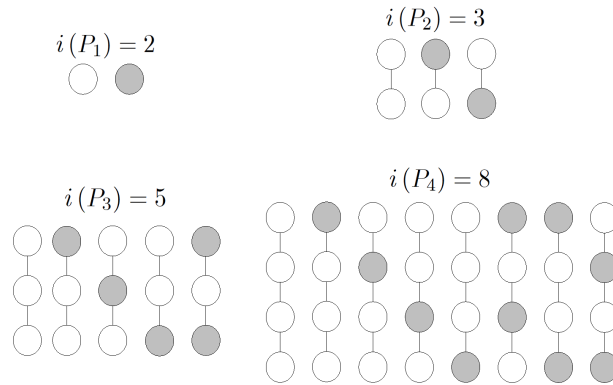
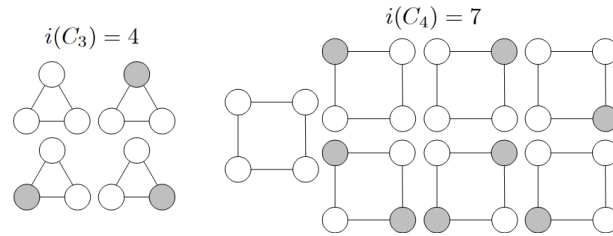
Mathematics Subject Classification : 05C69

1. Introduction

Given a graph $G = (V, E)$, a set $S \subseteq V$ is an independent set of vertices if no two vertices in S are adjacent. In our illustrations, we indicate membership in an independent set S by shading the vertices in S . Let the set of all independent sets of a graph G be denoted by $I(G)$ and let $i(G) = |I(G)|$. Note that $\emptyset \in I(G)$. The *path graph*, P_n , consists of the vertex set $V = \{1, 2, \dots, n\}$ and the edge set $E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$. The *cycle graph*, C_n , is the path graph, P_n , with the additional edge $\{1, n\}$.

Table 1 shows initial Fibonacci and Lucas numbers. In 1982, Prodinger and Tichy defined the Fibonacci number of a graph G , $i(G)$, to be the number of independent sets (including the empty

Received: 20 June 2014, Revised: 05 August 2014, Accepted: 15 September 2014.


 Figure 1. Independent sets of P_1 ; P_2 ; P_3 ; and P_4 .

 Figure 2. Independent sets of C_3 and C_4 .

set) of the graph G [5]. They did so because the Fibonacci number of the path graph P_n is the Fibonacci number F_{n+2} , and the Fibonacci number of the cycle graph C_n is the Lucas number L_n .

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

Table 1: Initial values of the Fibonacci and Lucas sequences

In [1], the authors of this paper use these graphs to combinatorially derive identities relating Fibonacci and Lucas numbers.

Example 1. $L_n = F_{n-1} + F_{n+1}$ for positive integers $n \geq 3$.

Proof. On the one hand we know that $i(C_n) = L_n$. On the other hand, vertex 1 is either a member of the independent set or it is not. If not, then any independent set from P_{n-1} , formed by vertices 2 through n , can be selected in $i(P_{n-1})$ ways. If in the set, then the remaining members can be selected in $i(P_{n-3})$ ways from the path formed by vertices 3 through $n-1$, since vertices 2 and n can not be selected. Hence, $L_n = i(C_n) = i(P_{n-3}) + i(P_{n-1}) = F_{n-1} + F_{n+1}$. \square

The Fibonacci sequence and the Lucas sequence are famous examples of the more general form called the Gibonacci sequence [3]. For integers $G_0 = a$ and $G_1 = b$, the Gibonacci sequence is defined recursively as $G_n = G_{n-1} + G_{n-2}$ for positive integers $n \geq 2$. Do other graphs exist whose Fibonacci numbers form a Gibonacci sequence?

The tadpole graph, $T_{n,k}$, is the graph created by concatenating C_n and P_k with an edge from any vertex of C_n to a pendent of P_k for integers $n \geq 3$ and $k \geq 0$. For ease of reference we label the vertices of the cycle c_1, \dots, c_n , the vertices of the path p_1, \dots, p_k where c_1 is adjacent to p_1 .

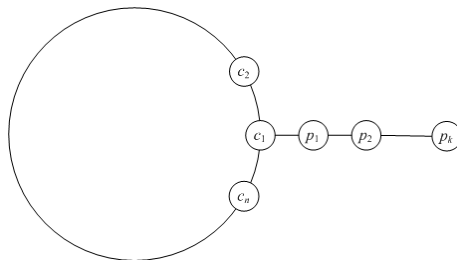


Figure 3. The Tadpole Graph $T_{n,k}$.

Example 2. Independent sets on $T_{3,2}$

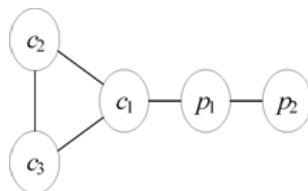


Figure 4. $T_{3,2}$.

$$I(T_{3,2}) = \{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{p_1\}, \{p_2\}, \{c_1, p_2\}, \{c_2, p_1\}, \{c_2, p_2\}, \{c_3, p_1\}, \{c_3, p_2\}\}$$

Theorem 1.1. $i(T_{n,k}) = i(T_{n,k-1}) + i(T_{n,k-2})$.

Proof. We show that $I(T_{n,k}) = I(T_{n,k-1}) \cup I(T_{n,k-2})$ where $I(T_{n,k-1}) \cap I(T_{n,k-2}) = \emptyset$. Partition $I(T_{n,k})$ into two disjoint subsets: sets where p_k is shaded and sets where p_k is not shaded. For every independent set in $I(T_{n,k-2})$, add an unshaded vertex p_{k-1} followed by a shaded vertex p_k to the end of the path graph. For every independent set in $I(T_{n,k-1})$, add an unshaded vertex p_k to the end of the path graph. Therefore, $i(T_{n,k}) = i(T_{n,k-1}) + i(T_{n,k-2})$. \square

Theorem 1.2. $i(T_{n,k}) = i(T_{n-1,k}) + i(T_{n-2,k})$.

Proof. Again, we show that $I(T_{n,k}) = I(T_{n-1,k}) \cup I(T_{n-2,k})$ where $I(T_{n-1,k}) \cap I(T_{n-2,k}) = \emptyset$. Label any three consecutive vertices of $T_{n,k}$ of degree two from the cycle as $n-1$, n and 1 . For every independent set in $I(T_{n-2,k})$, if vertex 1 is shaded (then vertex $n-2$ is not shaded), insert a shaded vertex $n-1$ and an unshaded vertex n , thus creating all independent sets of $T_{n,k}$ that include both 1 and $n-1$. If vertex 1 is not shaded, then insert a shaded vertex n and unshaded vertex $n-1$ creating all independent sets where vertex n is shaded. For every independent set in $I(T_{n-1,k})$, insert an unshaded vertex n which finally creates all independent sets where either 1 or $n-1$ is shaded or none of $n-1$, n and 1 are shaded. Therefore, $i(T_{n,k}) = i(T_{n-1,k}) + i(T_{n-2,k})$. \square

It is immediate that $i(T_{n,0}) = L_n$ since $T_{n,0} \cong C_n$. Computing $i(T_{3,2}) = 11$ and $i(T_{4,1}) = 12$ allows us to effortlessly fill in the following table since by Theorems 1.1 and 1.2, every row and column forms a Gibonacci sequence.

	k	0	1	2	3	4	5	6	7	8	9	10
n												
3		4	7	11	18	29	47	76	123	199	322	521
4		7	12	19	31	50	81	131	212	343	555	898
5		11	19	30	49	79	128	207	335	542	877	1419
6		18	31	49	80	129	209	338	547	885	1432	2317
7		29	50	79	129	208	337	545	882	1427	2309	3736
8		47	81	128	209	337	546	883	1429	2312	3741	6053
9		76	131	207	338	545	883	1428	2311	3739	6050	9789
10		123	212	335	547	882	1429	2311	3740	6051	9791	15842

Table 2: Fibonacci Numbers for the Tadpole Graph, $T_{n,k}$

It is easy to directly compute the Fibonacci number of any Tadpole graph.

Theorem 1.3. $i(T_{n,k}) = L_{n+k} + F_{n-3}F_k$.

Proof. We proceed with two base cases and strong induction on k . Suppose $k = 0$. Then $i(T_{n,0}) = i(C_n) = L_n + F_{n-3}F_0 = L_n$. For $k = 1$, combinatorially, $i(T_{n,1}) = i(C_n) + i(P_{n-1}) = L_n + F_{n+1}$. Now algebraically,

$$\begin{aligned}
 L_n + F_{n+1} &= L_{n+1} - L_{n-1} + F_n + F_{n-1} \\
 &= L_{n+1} - F_{n-2} + F_{n-1} \\
 &= L_{n+1} + F_{n-3}F_1.
 \end{aligned}$$

Finally, for general $k \geq 2$,

$$\begin{aligned}
 i(T_{n,k+1}) &= i(T_{n,k}) + i(T_{n,k-1}) \\
 &= L_{n+k} + F_{n-3}F_k + L_{n+k-1} + F_{n-3}F_{k-1} \\
 &= L_{n+k+1} + F_{n-3}F_{k+1}.
 \end{aligned}$$

□

Theorem 1.4. For $n \geq 3$ and $k \geq 0$,

1. $L_{n+k} = F_{n-1}F_{k+1} + F_{n+1}F_{k+2} - F_{n-3}F_k$;
2. $L_{n+k} = F_{n+1}F_k + L_nF_{k+1} - F_{n-3}F_k$;
3. $L_{n+k} = F_{n-1}F_{k+2} + F_{n+k+1} - F_{n-3}F_k$.

Proof. For 1, we know that there are $L_{n+k} + F_{n-3}F_k$ independent sets on the tadpole graph $T_{n,k}$. Now we partition $I(T_{n,k})$ into two disjoint sets: sets that contain c_1 and sets that do not. If c_1 is included in the independent set then c_2, c_n and p_1 are not. Hence, there are $i(P_{n-3})i(P_{k-1}) = F_{n-1}F_{k+1}$ such sets. If c_1 is not included in the independent set then there are $i(P_{n-1})i(P_k) = F_{n+1}F_{k+2}$ such sets. So, $L_{n+k} + F_{n-3}F_k = F_{n-1}F_{k+1} + F_{n+1}F_{k+2}$ and the result follows.

For 2, we partition $I(T_{n,k})$ into two disjoint sets: sets that contain p_1 and sets that do not. If p_1 is included in the independent set then c_1 and p_2 are not. Hence, there are $i(P_{n-1})i(P_{k-2}) = F_{n+1}F_k$ such sets. If p_1 is not included in the independent set then there are $i(C_n)i(P_{k-1}) = L_nF_{k+1}$ such sets. So $L_{n+k} + F_{n-3}F_k = F_{n+1}F_k + L_nF_{k+1}$ and the result follows.

For 3, we partition $I(T_{n,k})$ into two disjoint sets: sets that contain c_n and sets that do not. If c_n is included in the independent set then c_1 and c_{n-1} are not. Hence, there are $i(P_{n-3})i(P_k) = F_{n-1}F_{k+2}$ such sets. If c_n is not included in the independent set then there are $i(P_{n-1+k}) = F_{n+k+1}$ such sets. So, $L_{n+k} + F_{n-3}F_k = F_{n-1}F_{k+2} + F_{n+k+1}$ and the result follows. \square

2. Tadpole Triangle

We turn Table 2 into a triangular array where the (n, k) entry for $n \geq 3$ and $k \geq 0$ will be denoted $t_{n,k}$. Row n will represent the class of tadpole graphs with a total of n vertices. As the value of k increases by 1 through each row of the triangle, the cycle subgraph shrinks by one vertex and the length of the path subgraph increases by one. Thus, $t_{n,k}$ represents the number of independent sets on the Tadpole graph with n vertices with a path of length k (and thus, a cycle of length $n - k$). Hence, $t_{n,k} = i(T_{n-k,k})$. By Theorem 1.3, $t_{n,k} = L_n + F_{n-k-3}F_k$.

				4					
				7		7			
			11		12		11		
		18		19		19		18	
	29		31		30		31		29
47		50		49		49		50	47
76	81		79		80		79	81	76

Table 3: The Triangular Array of Fibonacci Numbers of the Tadpole Graph

As noted before, $t_{n,0} = i(T_{n,0}) = L_n$. Casual observation seems to indicate the rows the tadpole triangle are symmetric.

Theorem 2.1. $t_{n,k} = t_{n,n-k-3}$

Proof. Theorem 1.3 provides a quick, algebraic proof of the symmetry of rows since $t_{n,n-k-3} = i(T_{k+3,n-k-3}) = L_n + F_kF_{n-k-3} = i(T_{n-k,k}) = t_{n,k}$. \square

Proof. For a combinatorial proof of the symmetry in rows, consider c_2 in both $T_{k+3,n-k-3}$ and $T_{n-k,k}$. As before, we partition the tadpole graphs into two disjoint sets: those that contain c_2 and those that do not. Both tadpole graphs contain n vertices. Thus, the number of independent sets that do not contain c_2 in each tadpole graph is the number of independent sets on the path with

$n - 1$ vertices. Independent sets that contain c_2 , do not contain c_1 . This decomposes the tadpole graph into two disjoint paths. For both tadpole graphs, disjoint paths of length k and $n - k - 3$ are created. Both tadpole graphs lead to the same decomposition and $t_{n,k} = t_{n,n-k-3}$. \square

Theorem 2.2. $t_{n,k+1} - t_{n,k} = (-1)^k F_{n-2k-4}$ for $0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$.

Proof. Algebraically,

$$\begin{aligned} t_{n,k+1} - t_{n,k} &= L_n + F_{n-k-4}F_{k+1} - (L_n + F_{n-k-3}F_k) \\ &= F_{n-k-4}F_{k+1} - F_{n-k-3}F_k \\ &= (-1)^k F_{n-2k-4} \text{ by d'Ocagne's Identity.} \end{aligned}$$

\square

Proof. Combinatorially we proceed by initially considering the mapping

$$\Psi(S) = \begin{cases} S & \text{for } \{c_2, c_{n-k}\} \not\subseteq S \\ (S \setminus \{c_2\}) \cup \{c_1\} & \text{for } \{c_2, c_{n-k}\} \subseteq S \end{cases}$$

from $I(T_{n-k,k})$ to $I(T_{n-k-1,k+1})$ as illustrated in Figure 5. The identity mapping pairs together most independent sets but encounters obvious problems since independent sets in $I(T_{n-k,k})$ that contain both c_2 and c_{n-k} do not map to $I(T_{n-k-1,k+1})$, and independent sets in $I(T_{n-k-1,k+1})$ that contain both c_1 and c_{n-k} have no pre-image in $I(T_{n-k,k})$. If c_2 and c_{n-k} are both in the independent set, then remove c_2 from the independent set while including c_1 to create an independent set in $I(T_{n-k-1,k+1})$ to upgrade the identity mapping to $\Psi(S)$. We now have two subtle issues which provide the value of $t_{n,k+1} - t_{n,k}$. Independent sets in $I(T_{n-k,k})$ that contain the subset $\{p_1, c_2, c_{n-k}\}$ have no image. There are $i(P_{n-k-5})i(P_{k-2}) = F_{n-k-3}F_k$ such sets. Independent sets in $I(T_{n-k-1,k+1})$ that contain the subset $\{c_1, c_2, c_{n-k}\}$ have no pre-image. There are $i(P_{n-k-6})i(P_{k-1}) = F_{n-k-4}F_{k+1}$ such sets. Once again, $t_{n,k+1} - t_{n,k} = F_{n-k-4}F_{k+1} - F_{n-k-3}F_k = (-1)^k F_{n-2k-4}$. \square

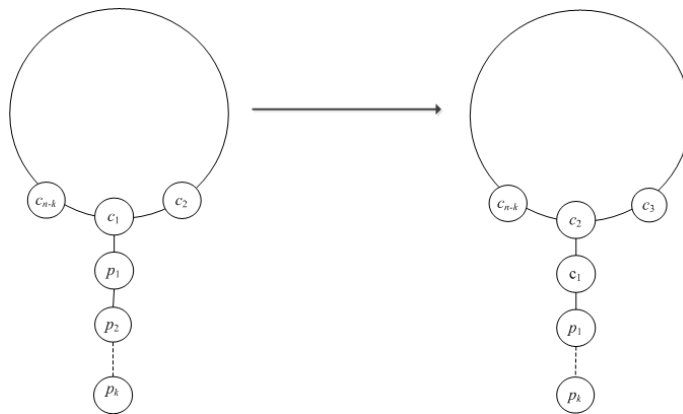


Figure 5. Mapping $I(T_{n-k,k})$ to $I(T_{n-k-1,k+1})$.

Theorem 2.3. $\sum_{k=0}^{n-3} (-1)^k t_{n,k} = \begin{cases} 0 & \text{for even } n \\ 2F_n & \text{for odd } n. \end{cases}$

Proof. For even n the result is trivial due to the symmetry of row values. For odd n , we proceed by induction. Base cases abound from Table 3. Assume n is odd and $\sum_{k=0}^{n-3} (-1)^k t_{n,k} = 2F_n$. Moving on to the next odd value we consider

$$\begin{aligned}
 \sum_{k=0}^{n-1} (-1)^k t_{n+2,k} &= \left(\sum_{k=0}^{n-3} (-1)^k t_{n+2,k} \right) + (-1)^{n-2} t_{n+2,n-2} + (-1)^{n-1} t_{n+2,n-1} \\
 &= \sum_{k=0}^{n-3} (-1)^k t_{n+2,k} - t_{n+2,n-2} + L_{n+2} \\
 &= \left(\sum_{k=0}^{n-3} (-1)^k [t_{n,k} + t_{n+1,k}] \right) - L_{n+2} + F_{n+2-4} + L_{n+2} \\
 &= \sum_{k=0}^{n-3} (-1)^k t_{n,k} + (-1)^k t_{n+1,k} - L_{n+2} - F_{n-2} + L_{n+2} \\
 &= \left(\sum_{k=0}^{n-3} (-1)^k t_{n,k} \right) + \left(\sum_{k=0}^{n-2} (-1)^k t_{n+1,k} \right) + t_{n+1,n-2} - L_{n+2} - F_{n-2} + L_{n+2} \\
 &= 2F_n + 0 + t_{n+1,n-2} - F_{n-2} = 2F_n + L_{n+1} - F_{n-2} \\
 &= 3F_n + F_{n+2} - F_{n-2} = 2F_n + F_{n-1} + F_{n+2} \\
 &= F_n + F_{n+1} + F_{n+2} = 2F_{n+2}
 \end{aligned}$$

□

Theorem 2.4. The ratio of consecutive row sums converges to the golden ratio ϕ .

Proof. The sum of row n can be written as

$$\begin{aligned}
 \sum_{k=0}^{n-3} t_{n,k} &= \sum_{k=0}^{n-3} (L_n + F_{n-k-3} F_k) \\
 &= \sum_{k=0}^{n-3} L_n + \sum_{k=0}^{n-3} F_{n-k-3} F_k \\
 &= (n-2)L_n + \frac{(n-3)L_{n-3} - F_{n-3}}{5}
 \end{aligned}$$

Now,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{(n+1-2)L_{n+1} + \frac{(n+1-3)L_{n+1-3} - F_{n+1-3}}{5}}{(n-2)L_n + \frac{(n-3)L_{n-3} - F_{n-3}}{5}} \\
 &= \lim_{n \rightarrow \infty} \frac{(n-1)L_{n+1} + \frac{(n-2)L_{n-2} - F_{n-2}}{5}}{(n-2)L_n + \frac{(n-3)L_{n-3} - F_{n-3}}{5}} \\
 &= \lim_{n \rightarrow \infty} \frac{nL_{n+1} + nL_{n-2} - F_{n-2}}{nL_n + nL_{n-3} - F_{n-3}} \\
 &= \lim_{n \rightarrow \infty} \frac{L_{n+1} + L_{n-2}}{L_n + L_{n-3}} \\
 &= \lim_{n \rightarrow \infty} \frac{L_n}{L_{n-1}} = \phi.
 \end{aligned}$$

□

A perfect matching (or 1-factor) in a graph $G = (V, E)$ is a subset S of edges of E such that every vertex in V is incident to exactly one edge in S . In [2], Gutman and Cyvin define the L-shaped graph, $L_{p,q}$, to be the graph with $p+q+1$ copies of C_4 as illustrated in Figure 6 by $L_{2,1}$.

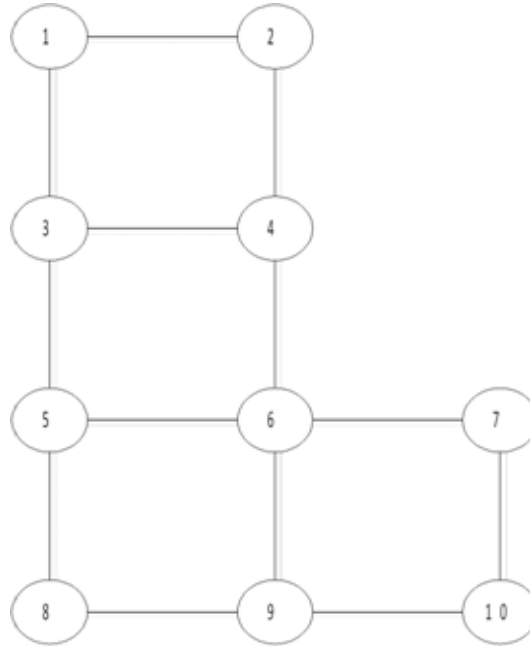


Figure 6. $L_{2,1}$.

$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 10\}, \{8, 9\}\}$
$\{\{1, 2\}, \{3, 4\}, \{5, 8\}, \{6, 9\}, \{7, 10\}\}$
$\{\{1, 2\}, \{3, 5\}, \{4, 6\}, \{7, 10\}, \{8, 9\}\}$
$\{\{1, 2\}, \{3, 4\}, \{5, 8\}, \{6, 7\}, \{9, 10\}\}$
$\{\{1, 3\}, \{2, 4\}, \{5, 6\}, \{8, 9\}, \{7, 10\}\}$
$\{\{1, 3\}, \{2, 4\}, \{5, 8\}, \{6, 7\}, \{9, 10\}\}$
$\{\{1, 3\}, \{2, 4\}, \{5, 8\}, \{6, 9\}, \{7, 10\}\}$

 Table 4: The seven perfect matchings of $L_{2,1}$

They show that the number of perfect matchings in $L_{p,q}$ is $F_{p+q+2} + F_{p+1}F_{q+1}$. These values correspond to the columns of the tadpole triangle. This correspondence provides a quick proof of the symmetry of rows of the tadpole triangle since $L_{p,q} \approx L_{q,p}$. We number columns starting with the center at $i = 0$.

Theorem 2.5. *The number of perfect matchings in $L_{p,q}$ is given by $t_{p+q+1,q-1}$, the p^{th} entry in columns $\pm(p - q)$.*

Proof. Since the tadpole triangle is symmetric we can assume that $p \geq q$. By Theorem 1.3,

$$\begin{aligned}
 t_{p+q+1,q-1} &= L_{p+q+1} + F_{p-1}F_{q-1} \\
 &= F_{p+q+2} + F_{p+q} + F_{p-1}F_{q-1} \\
 &= F_{p+q+2} + (F_{p+1}F_{q+1} - F_{p-1}F_{q-1}) + F_{p-1}F_{q-1} \\
 &= F_{p+q+2} + F_{p+1}F_{q+1}.
 \end{aligned}$$

□

3. Future Work

In [4], Pederson and Vestergaard show that for every unicyclic graph G of order n , $L_n \leq i(G) \leq 3 \times 2^{n-3} + 1$. Furthermore they show that the minimum bound is realized only for $T_{n,0} \approx C_n$ and $T_{3,n-3}$. The maximum bound occurs only for C_4 and the graph with $n - 3$ pendants adjacent to the same vertex of C_3 . The technique of this paper can be used to precisely determine the Fibonacci number of many classes of unicyclic graphs.

Acknowledgement

The authors thank programs at Kennesaw State University and Brigham Young University for travel support when presenting this work at the Joint Mathematics Meetings in 2014 and the Forty-Fifth Southeastern International Conference on Combinatorics, Graph Theory, and Computing. At KSU, The Center for Excellence in Teaching and Learning's URCA program and at BYU, The Center for Undergraduate Research in Mathematics both encourage the inclusion of undergraduate students in faculty research and dissemination of results at professional conferences.

References

- [1] J. DeMaio and J. Jacobson, Fibonacci and Lucas identities via graphs, *Springer Proceedings in Mathematics & Statistics: Topics from the 8th Annual UNCG Regional Mathematics and Statistics Conference*, (pp. 85–92). Springer New York.
- [2] I. Gutman and S. Cyvin, A result on 1-factors related to Fibonacci numbers, *Fibonacci Quarterly* **28(1)** (1990), 81–84.
- [3] T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley-Interscience, New York, 2001.
- [4] A.S. Pederson and P.D. Vestergaard, The number of independent sets in unicyclic graphs, *Discrete Applied Mathematics* **152** (2005), 246–256
- [5] H. Prodinger and R. Tichy, Fibonacci numbers of graphs, *Fibonacci Quarterly* **20(1)** (1982), 16–21.